Math 245C Lecture 6 Notes

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1 The Marcinkiewicz Interpolation Theorem (cont.)[2]

Today's lecture was given by a guest lecturer, Alpár Mészáros.

1.1 Conclusion of the proof

Last time, we were proving the following theorem.

Theorem 1.1 (Marcinkiewicz[1] interpolation theorem). *Let $\mathcal{F}$ be the set of measurable functions on Y. Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ be real numbers such that $p_0 \le q_0$, $p_1 \le q_1$, and $q_0 \neq q_1$. Let $t \in (0,1)$, and let p, q be defined as*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Assume that $T : L^{p_0}(\mu) + L^{p_1}(\mu) \to \mathcal{F}$ be sublinear and of weak type (q_0, p_0) and (q_1, p_1) (there are $c_0, c_1 > 0$ such that if $q_0, q_1 \neq \infty$, $(\alpha^{q_0} \lambda_{T(f)})^{1/q_0} \le c_0 \|f\|_{p_0}$ and $(\alpha^{q_1} \lambda_{T(f)})^{1/q_1} \le c_1 \|f\|_{p_1}$). Then the following hold:

1. *T is strong type (p, q) (there exists $B_p > 0$ such that $\|Tf\|_q \le B_p \|f\|_p$ for all $f \in L^p(\mu)$).*

2. *If $p_0 < \infty$, then $\lim_{p \to p_0} B_p |p_0 - p| < \infty$. If $p_1 < \infty$, then $\lim_{p \to p_1} B_p |p_1 - p| < \infty$. If $p_0 = \infty$, (B_p) remains bounded as $p \to p_0$. If $p_1 = \infty$, (B_p) remains bounded as $p \to p_1$.*

Proof. Without loss of generality we can assume $p_0 < p_1$. We showed that

$$\|Tf\|_q^q \le \sum_{j=0}^{1} 2^q q C_j^{q_j} p^{q_j/p_j} \int_0^\infty \left(\int_0^\infty \phi_j(\alpha, \beta)\, d\beta \right)^{q_j/p_j} d\alpha.$$

[1] Marcinkiewicz was a Polish mathematician who died during WWII. Zygmund discovered afterwards that he proved this result and gave credit to Marcinkiewicz.

Here,

$$\phi_j(\alpha,\beta) := \mathbb{1}_j(\alpha,\beta)\beta^{p_j-1}\lambda_f(\beta)\alpha^{(q-q_j-1)p_j/q_j},$$

where $\mathbb{1}_0$ is the indicator of $\{(\alpha,\beta) : \beta > A\}$ and $\mathbb{1}_1$ is the indicator of $\{(\alpha,\beta) : \beta < A\}$. We want to set $A = \alpha^\sigma$ for some good choice of σ. Look at the term with ϕ_0.

Case 1: $\sigma > 0$: If $\beta > \alpha^\sigma$, then $\alpha < \beta^{1/\sigma}$. After Minkowski's inequality,

$$\int_0^\infty \left(\int_0^{\beta^{1/\sigma}} \alpha^{q-q_0-1}\, d\alpha\right)^{p_0/q_0} \beta^{p_0-1}\lambda_f(\beta)\, d\beta$$

$$= \int_0^\infty \left(\frac{1}{q-q_0}\right)^{p_0/q_0} \beta^{(q-q_0)p_0/(q_0\sigma)}\beta^{p_0-1}\lambda_f(\beta)\, d\beta$$

Now pick

$$\sigma = \frac{p_0}{q_0}\frac{q_0-q}{p_0-p} > 0.$$

Since we want this to be positive, we need to assume that $q_0 < q_1$. The previous quantity becomes

$$\left(\frac{1}{q-q_0}\right)^{p_0/q_0} p^{-1}\|f\|_p^p.$$

If $q_0 > q_1$, then $\sigma < 0$. So $\beta > \alpha \implies \alpha > \beta^{1/\sigma}$. Then what changes is the integral becomes an integral $\int_0^\infty \int_{\beta^{1/\sigma}}^\infty$. We get

$$\int_0^\infty \frac{1}{q-q_0}\left([\alpha^{q-q_0}]_{\beta^{1/\sigma}}^\infty\right)^{p_0/q_0} d\beta = \left(\frac{1}{q_0-q}\right)^{p_0/q_0} p^{-1}\|f\|_p^p.$$

For the term involving ϕ_1, the computation is very similar with (p_1, q_1) instead of (p_0, q_0). The key property here is that

$$\sigma = \frac{p_0}{q_0}\frac{q_0-q}{p_0-p} = \frac{p_1}{q_1}\frac{q_1-q}{p_1-p},$$

which follows from the construction of p, q.

Remaining case 1: Assume that $p_1 = q_1 = \infty$. Then $\|Tf\|_\infty \le C_1\|f\|_\infty$ (because (∞,∞)-weak means (∞,∞)-strong). We have $\|Th_A\|_\infty \le C_1\|h_A\|_\infty$. We want to choose A in a way that ϕ_1 becomes 0. We claim that $A = \alpha/C_1$ works. In this case, $\beta < \alpha/C_1$. We get

$$\|Th_A\|_\infty \le C_1\|h_A\|_\infty \le C_1 A = C_1\frac{\alpha}{C_1} = \alpha.$$

We have

$$\mathbb{1}_{\{\beta<\alpha/C_1\}}\lambda_f(\beta) = \mathbb{1}_{\{\beta<\alpha/C_1\}}\lambda_f h_A(\beta) = 0,$$

so ϕ_1 does not give a contribution. Do the same computation with ϕ_0, replacing α^σ with α/C_1.

Remaining case 2: Assume $p_0 < p_1 < \infty$ and $q_0 < q_1 = \infty$. Choose A in a way such that $\lambda_{Th_A}(\beta) = 0$ ($\|Th_A\|_\infty \leq C_1 \|f\|_{p_1}$). If we choose $A = (\alpha/d)^\sigma$, where $\sigma = p_1/(p_1 - p)$ and $d = C_1[p_1\|f\|_p^p/p]^{1/p_1}$, we get

$$\|Th_A\| \leq \alpha.$$

Remaining case 3: If $p_0 < p_1 < \infty$ and $q_1 < q_0 = \infty$, we want that $\lambda_{Tg_A}(\alpha) = 0$. In this case, choose A such that $A = (\alpha/d)^\sigma$.

We have obtained that

$$\|Tf\|_q^q \leq \text{constant } \|f\|_p^p.$$

Define B_p such that $\sup\{\|Tf\|_q : \|f\|_p = 1\} \leq B_p$. You can write down the constant explicitly in all cases. □

1.2 L^p-estimates for the Hardy-Littlewood Maximal function

For $f \in L^1_{\text{loc}}$, let

$$(Hf)(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, dy$$

be the Hardy-Littlewood maximal function. Then H is sublinear. H is (∞, ∞)-strong type. We can show that H is $(1,1)$-weak type. By the Marcinkiewicz interpolation theorem, we get that

$$\|Hf\|_p \leq C(n)\frac{p}{p-1}\|f\|_p$$

for any $p \in (1, \infty]$.

However, H is not $(1,1)$-strong type. Come up with an example as an exercise.